

A CRITERION OF NORMALITY BASED ON A SINGLE HOLOMORPHIC FUNCTION II

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ABSTRACT. In this paper, we continue to discuss normality based on a single holomorphic function. We obtain the following result. Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$. Let $k \geq 2$ be an integer and let $h(\not\equiv 0)$ be a holomorphic function on D , such that $h(z)$ has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$: (a) $f(z) = 0 \implies f'(z) = h(z)$ and (b) $f'(z) = h(z) \implies |f^{(k)}(z)| \leq c$, where c is a constant. Then \mathcal{F} is normal on D .

A geometrical approach is used to arrive at the result which significantly improves the previous results of the authors, *A criterion of normality based on a single holomorphic function*, Acta Math. Sinica, English Series (1) **27** (2011), 141–154 and of Chang, Fang, and Zalcman, *Normal families of holomorphic functions*, Illinois Math. J. (1) **48** (2004), 319–337. We also deal with two other similar criteria of normality. Our results are shown to be sharp.

1. INTRODUCTION

In [11], X.C. Pang and L. Zalcman proved the following theorem.

Theorem PZ. *Let \mathcal{F} be a family of meromorphic functions on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , where $k \geq 1$ is an integer. Suppose there exist constants $b \neq 0$ and $h > 0$ such that, for every $f \in \mathcal{F}$, $f(z) = 0 \iff f^{(k)}(z) = b$ and $f(z) = 0 \implies 0 < |f^{(k+1)}(z)| \leq h$. Then \mathcal{F} is a normal family on D .*

Then, in [1], J.M Chang, M.L. Fang, and L. Zalcman proved the following result.

Theorem CFZ1. [1, Theorem 4] *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$. Let $k \geq 2$ be an integer, and let $h(z) \neq 0$ be a function analytic in D . Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

- (a) $f(z) = 0 \implies f'(z) = h(z)$; and

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(b) $f'(z) = h(z) \implies |f^{(k)}(z)| \leq c$, where c is a constant.

Then \mathcal{F} is normal on D .

And in [4], we replaced the condition $h(z) \neq 0$ with $h(z) \not\equiv 0$ and obtained the following result.

Theorem LN. *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$. Let $k \geq 2$ be an integer, and let $h(z) (\not\equiv 0)$ be a holomorphic function on D , all of whose zeros have multiplicity at most $k - 1$, that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

- (a) $f(z) = 0 \implies f'(z) = h(z)$ and
- (b) $f'(z) = h(z) \implies |f^{(k)}(z)| \leq c$, where c is a constant.

Then \mathcal{F} is normal on D .

We now pose the following question: can the restriction for the zeros of $h(z)$ with multiplicity at most $k - 1$ be dropped? In this paper, we continue to study the above problem and obtain an affirmative answer.

Theorem 1. *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$. Let $k \geq 2$ be an integer, and let $h(z) (\not\equiv 0)$ be a holomorphic function on D that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

- (a) $f(z) = 0 \implies f'(z) = h(z)$ and
- (b) $f'(z) = h(z) \implies |f^{(k)}(z)| \leq c$, where c is a constant.

Then \mathcal{F} is normal on D .

Also in [1], the case for the k -th derivative was considered and the following result was proved .

Theorem CFZ2. [1, Theorem 1] *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , where $k \neq 2$ is a positive integer; and let $h(z) \neq 0$ be a function analytic in D . Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

- (a) $f(z) = 0 \implies f^{(k)}(z) = h(z)$; and

(b) $f^{(k)}(z) = h(z) \implies |f^{(k+1)}(z)| \leq c$, where c is a constant.

Then \mathcal{F} is normal on D .

For the case $k = 2$, the following result was obtained.

Theorem CFZ3. [1, Theorem 3] *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros are multiple, where $s \geq 4$ is an even integer; and let $h(z) \neq 0$ be a function analytic in D . Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \implies f''(z) = h(z)$; and

(b) $f''(z) = h(z) \implies |f'''(z)| + |f^{(s)}(z)| \leq c$, where c is a constant.

Then \mathcal{F} is normal on D .

In view of the improvement of Theorems CFZ1 and LN via Theorem 1, the question that naturally arises concerning Theorem CFZ2 and CFZ3, is whether the condition $h(z) \neq 0$, $z \in D$, can be relaxed to “ $h \not\equiv 0$ ”. It turns out that the answer is negative in both cases. It is negative even if h has no common zero with any $f \in \mathcal{F}$ (like in Theorem 1). To construct the first example, concerning Theorem CFZ2, we first need to mention the following famous result of F. Lucas.

Theorem Lu. [5], [6, p. 22] *Let $P(z)$ be a nonconstant polynomial. Then all the zeros of $P'(z)$ lie in the convex hull H of the zeros of $P(z)$. Moreover, there are no zeros of $P'(z)$ on the boundary of H , unless this zero is a multiple zero of $P(z)$ or the zeros of $P(z)$ are colinear.*

Example 1. Let $r \geq 1$ and $k \geq 3$ be integers, $D = \Delta$ be the unit disc and $h(z) = z^r$. Define

$$f_n(z) = a_n \left(z^\ell - \frac{1}{n^\ell} \right)^k,$$

where $\ell = k + r$ and $a_n = \frac{n^{(k-1)\ell}}{k!\ell^k}$.

We have

$$f_n(z) = a_n \prod_{j=1}^{\ell} \left(z - \alpha_j^{(n)} \right)^k,$$

where $\alpha_j^{(n)} = \frac{\exp\left(i\frac{2\pi j}{\ell}\right)}{n}$, for $1 \leq j \leq \ell$.

By calculation,

$$\begin{aligned} f_n^{(k)}(\alpha_j^{(n)}) &= k!a_n \prod_{t=1, t \neq j}^{\ell} (\alpha_j^{(n)} - \alpha_t^{(n)})^k = k!a_n \left[\left(z^\ell - \frac{1}{n^\ell} \right)' \Big|_{z=\alpha_j^{(n)}} \right]^k \\ &= k!a_n \ell^k (\alpha_j^{(n)})^{k(\ell-1)}. \end{aligned}$$

Thus,

$$(1) \quad \arg \left[f_n^{(k)}(\alpha_j^{(n)}) \right] = (\ell-1)k \cdot \frac{2\pi j}{\ell} = -\frac{2\pi k j}{\ell} = \frac{2\pi r i}{\ell} = \arg \left[z^r \Big|_{z=\alpha_j^{(n)}} \right].$$

Here the equalities are modulo 2π , and we used in the last equality that $r+k=\ell$.

We have

$$(2) \quad \left| f_n^{(k)}(\alpha_j^{(n)}) \right| = \frac{k! \ell^k n^{\ell(k-1)}}{k! \ell^k} \left(\frac{1}{n} \right)^{k(\ell-1)} = \left(\frac{1}{n} \right)^r = |z^r| \Big|_{z=\alpha_j^{(n)}}.$$

From (1) and (2) we have that $f_n(z) = 0 \implies f_n^{(k)}(z) = h(z)$, i.e., assumption (a) of Theorem CFZ2 holds.

In order to confirm (b) of Theorem CFZ2, set

$$\tilde{f}_n(z) = f_n(z) - \frac{z^\ell}{\ell(\ell-1) \cdots (r+1)}.$$

We have $f_n^{(k)}(z) = h(z) \iff \tilde{f}_n^{(k)}(z) = 0$.

Now

$$(3) \quad \tilde{f}_n(z) = 0 \iff \frac{n^{k(\ell-1)-r}}{k! \ell^k} \left(z^\ell - \frac{1}{n^\ell} \right)^k = \frac{z^\ell}{\ell(\ell-1) \cdots (r+1)}.$$

Suppose by negation that there exist a sequence $\{z_n\}_{n=1}^\infty$ ($z_n \rightarrow 0$) and a sequence of natural numbers $\{k_n\}_{n=1}^\infty$ ($k_n \xrightarrow{n \rightarrow \infty} \infty$), such that $\tilde{f}_{k_n}(z_n) = 0$. Then since

$$\frac{(k_n z_n)^\ell - 1}{(k_n z_n)^\ell} \xrightarrow{n \rightarrow \infty} 1, \text{ from (3) we get}$$

$$(4) \quad \frac{k_n^{(k-1)\ell} (k_n z_n)^{k\ell}}{k_n^{k\ell} z_n^\ell} \xrightarrow{n \rightarrow \infty} \frac{k! \ell^k}{\ell(\ell-1) \cdots (r+1)}.$$

But the left hand side of (4) tends to ∞ , as $n \rightarrow \infty$, a contradiction.

We deduce that there exists some $0 < C_1 < \infty$, such that every zero z_n of \tilde{f}_n satisfies $|z_n| \leq \frac{C_1}{n}$. By Theorem Lu, we have also $|\hat{z}_n| \leq \frac{C_1}{n}$ for every \hat{z}_n , which is a zero of $\tilde{f}_n^{(k)}$. But those $\{\hat{z}_n\}$ are exactly the points where $f_n^{(k)}(z) = h(z)$.

Hence $f_n^{(k)}(z) = h(z)$ implies that $|z| \leq \frac{C_1}{n}$, and we have only to prove the following claim.

Claim 1. *There exists $0 < C < \infty$, such that $|z| \leq \frac{C_1}{n}$ implies $|f_n^{(k+1)}(z)| \leq C$.*

Proof. We have $f_n(z) = \frac{n^{(k-1)\ell}}{k!\ell^k} \left(z^\ell - \frac{1}{n^\ell}\right)^k = \frac{n^{(k-1)\ell}}{k!\ell^k} \sum_{j=0}^k \binom{k}{j} z^{\ell j} \left(\frac{1}{n}\right)^{\ell(k-j)} (-1)^{k-j}$. Thus, since $\ell j \geq k+1$ only for $j \geq 1$, we get that

$$f_n^{(k+1)}(z) = \frac{n^{(k-1)\ell}}{k!\ell^k} \sum_{j=1}^k \binom{k}{j} \left(\frac{1}{n}\right)^{\ell k - \ell j} (-1)^{k-j} \ell j (\ell j - 1) \cdots (\ell j - k + 1) z^{\ell j - k - 1}.$$

Thus, if $|z| \leq \frac{C_1}{n}$, then

$$\begin{aligned} |f_n^{(k+1)}(z)| &\leq \frac{n^{(k-1)\ell}}{k!\ell^k} \sum_{j=1}^k \binom{k}{j} C_1^{\ell j - k - 1} \ell j (\ell j - 1) \cdots (\ell j - k + 1) n^{k+1-\ell j} \cdot n^{\ell j - \ell k} \\ &= \frac{n^{k+1-\ell}}{k!\ell^k} \sum_{j=1}^k \binom{k}{j} C_1^{\ell j - k - 1} \ell j (\ell j - 1) \cdots (\ell j - k + 1) \leq C, \end{aligned}$$

where $C = \frac{1}{k!\ell^k} \sum_{j=1}^k \binom{k}{j} C_1^{\ell j - k - 1} \ell j (\ell j - 1) \cdots (\ell j - k + 1)$. (Here we used that $k+1-\ell \leq 0$.)

The Claim is proved. \square

Hence, $\{f_n\}$ with h satisfy (a) and (b) of Theorem CFZ2, but $\{f_n\}$ is not normal at $z = 0$.

Observe that when $k = 1$, then $a_n = \frac{1}{\ell} \not\rightarrow \infty$, and we do not get a non-normal family, as expected by Theorem 1.

The following example shows that the condition $h(z) \neq 0$ is essential also for Theorem CFZ3.

Example 2. (cf. [1, Ex. 4]) Let $s \geq 4$ be an even integer and consider the family $\mathcal{F} = \{f_n(z)\}_{n=1}^\infty$,

$$f_n(z) = \frac{n^s}{2s^2} \left(z^s - \frac{1}{n^s}\right)^2 \quad \text{on } \Delta.$$

Let $h(z) = z^{s-2}$.

We have that

$$f_n(z) = \frac{n^s}{2s^2} \prod_{j=1}^s \left(z - \alpha_j^{(n)}\right)^2,$$

where $\alpha_j^{(n)} = \frac{\exp(i2\pi j/s)}{n}$, $1 \leq j \leq s$.

By calculation we have

$$(5) \quad f_n''(z) = \frac{n^s}{s} \left((2s-1)z^s - \frac{(s-1)}{n^s} \right) z^{s-2},$$

$$(6) \quad \begin{aligned} f_n'''(z) &= \frac{n^s}{s} \left[(2s-1)(2s-2)z^s - \frac{(s-1)(s-2)}{n^s} \right] z^{s-3} \\ &= \frac{n^s}{s} (s-1)z^{s-3} \left[(4s-2)z^s - \frac{s-2}{n^s} \right], \end{aligned}$$

and

$$(7) \quad f_n^{(s)}(z) = \frac{n^s}{s} \left[(2s-1)(2s-2) \cdots (s+1)z^s - \frac{(s-1)!}{n^s} \right].$$

Now, if $f_n(z) = 0$, then $z = \alpha_j^{(n)}$ for some $1 \leq j \leq s$, and thus $z^s = \frac{1}{n^s}$ and by (5), $f_n''(z) = z^{s-2} = h(z)$.

If $f_n''(z) = z^{s-2} = h(z)$, then by (5), $z = 0$ or $z = \alpha_j^{(n)}$, $1 \leq j \leq s$. By (6) and (7), we get

$$(8) \quad f_n^{(3)}(0) = 0, \quad f_n^{(s)}(0) = -\frac{(s-1)!}{n^s}$$

and

$$(9) \quad f_n^{(3)}(\alpha_j^{(n)}) = 3(s-1)\frac{1}{n^{s-3}}, \quad f_n^{(s)}(\alpha_j^{(n)}) = \frac{1}{s} \left[\frac{(2s-1)!}{s!} - (s-1)! \right].$$

From (8) and (9), we see that the family \mathcal{F} with h satisfy assumption (a) and (b) of Theorem CFZ3, but \mathcal{F} is not normal at $z = 0$. Indeed, the reason must be that $h(0) = 0$.

In Example 1, we have that $f^{(k+1)}(z) \neq 0$ at the zero points of $f^{(k)}(z) - h(z)$. If we strengthen condition (b) of Theorem CFZ2 to be $f^{(k)}(z) = h(z) \implies f^{(k+1)}(z) = 0$, then we can obtain the following normal criterion.

Theorem 2. *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , where $k \neq 2$ be a positive integer. Let $h(z) (\neq 0)$ be a holomorphic function on D , that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

- (a) $f(z) = 0 \implies f^{(k)}(z) = h(z)$; and
- (b) $f^{(k)}(z) = h(z) \implies f^{(k+1)}(z) = 0$.

Then \mathcal{F} is normal on D .

Similarly, if we strengthen the condition (b) of Theorem CFZ3 to $f''(z) = h(z) \implies f'''(z) = f^{(s)}(z) = 0$, then we can also obtain the normality criterion.

Theorem 3. *Let \mathcal{F} be a family of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros are multiple, where $s \geq 2$ is an even integer. Let $h(z) (\neq 0)$ be a holomorphic function on D , that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

- (a) $f(z) = 0 \implies f''(z) = h(z)$; and
- (b) $f''(z) = h(z) \implies f'''(z) = f^{(s)}(z) = 0$.

Then \mathcal{F} is normal on D .

Before we go to the proofs of the main results, let us set some notation. Throughout, D is a domain in \mathbb{C} . For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ and $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$. The unit disc will be denoted by Δ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We write $f_n(z) \xrightarrow{X} f(z)$ on D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric, uniformly on compact subsets of D , and $f_n \Rightarrow f$ on D if the convergence is in the Euclidean metric. For a meromorphic function $f(z)$ in D and $a \in \widehat{\mathbb{C}}$, $\overline{E}_f(a) := \{z \in D : f(z) = a\}$. The spherical derivative of the meromorphic function f at the point z is denoted by $f^\#(z)$.

Frequently, given a sequence $\{f_n\}_1^\infty$ of functions, we need to extract an appropriate subsequence; and this necessity may recur within a single proof. To avoid the awkwardness of multiple indices, we again denote the extracted subsequence by $\{f_n\}$ (rather than, say, $\{f_{n_k}\}$) and designate this operation by writing “taking a subsequence and renumbering,” or simply “renumbering”. The same convention applies to sequences of constants.

The plan of the paper is as follows. In Section 2, we state a number of preliminary results. Then in Section 3 we prove Theorem 1. Finally, in Section 4 we prove Theorem 2.

2. PRELIMINARY RESULTS

The following lemma is the local version of a well-known lemma of X. C. Pang and L. Zalcman [11, Lemma 2]. For a proof see [4, Lemma 2], also cf. [9, Lemma 2], [14, pp. 216–217], [7, pp. 299–300], [8, p. 4].

Lemma 1. *Let \mathcal{F} be a family of functions meromorphic in a domain D , all of whose zeros have multiplicity at least k , and suppose that there exists $A \geq 1$, such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if \mathcal{F} is not normal at $z_0 \in D$, there exist, for each $0 \leq \alpha \leq k$,*

- (a) *points $z_n \rightarrow z_0$;*
- (b) *functions $f_n \in \mathcal{F}$; and*
- (c) *positive numbers $\rho_n \rightarrow 0^+$*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + f_n \zeta) \xrightarrow{X} g(\zeta)$ on \mathbb{C} , where g is a nonconstant meromorphic function on \mathbb{C} , such that for every $\zeta \in \mathbb{C}$, $g^\#(\zeta) \leq g^\#(0) = kA + 1$.

Lemma 2. [1, Lemma 5] *Let f be a nonconstant entire function of order ρ , $0 \leq \rho \leq 1$, all of whose zeros have multiplicity at least k , where $k \neq 2$ is a positive integer. And let $a \neq 0$ be a constant. If $\overline{E}_f(0) \subset \overline{E}_{f^{(k)}}(a) \subset \overline{E}_{f^{(k+1)}}(0)$, then*

$$f(z) = \frac{a(z-b)^k}{k!},$$

where b is a constant.

Lemma 3. [1, Lemma 6] *Let f be a nonconstant entire function of order ρ , $0 \leq \rho \leq 1$, all of whose zeros are multiple. Let $s \geq 4$ be an even integer and $a \neq 0$ be a constant. If $\overline{E}_f(0) \subset \overline{E}_{f''}(a) \subset \overline{E}_{f'''}(0) \cap \overline{E}_{f^{(s)}}(0)$, then*

$$f(z) = \frac{a(z-b)^2}{2},$$

where b is a constant.

Lemma 4. (see [2, pp. 118–119, 122–123]) *Let f be a meromorphic function on \mathbb{C} . If $f^\#$ is uniformly bounded on \mathbb{C} , then the order of f is at most 2. If f is an entire function, then the order of f is at most 1.*

The following lemma is a slight generalization of Theorem CFZ2 for sequences.

Lemma 5. (cf. [4, Lemma 5]) *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , and let $\{h_n\}$ be a sequence of functions analytic on D such that $h_n(z) \Rightarrow h(z)$ on D , where $h(z) \neq 0$ for $z \in D$ and $k \neq 2$ be a positive integer. Suppose that, for each n , $f_n(z) = 0 \implies f_n^{(k)}(z) = h_n(z)$ and $f_n^{(k)}(z) = h_n(z) \implies f_n^{(k+1)}(z) = 0$. Then $\{f_n\}$ is normal on D .*

Proof. Suppose to the contrary that there exists $z_0 \in D$ such that $\{f_n\}$ is not normal at z_0 . The convergence of $\{h_n\}$ to h implies that, in some neighborhood of z_0 , we have $f_n(z) = 0 \Rightarrow |f_n^{(k)}(z)| \leq |h(z_0)| + 1$ (for large enough n). Thus we can apply Lemma 1 with $\alpha = k$ and A such that $kA + 1 > \max \left\{ |h(z_0)| + 1, \frac{|h(z_0)|}{(k-1)!}, \frac{k \cdot k!}{|h(z_0)|} \right\} = \max \left\{ |h(z_0)| + 1, \frac{k \cdot k!}{|h(z_0)|} \right\}$. So we can take an appropriate subsequence of $\{f_n\}$ (denoted also by $\{f_n\}$ after renumbering), together with points $z_n \rightarrow z_0$ and positive numbers $\rho_n \rightarrow 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} \xrightarrow{\chi} g(\zeta) \quad \text{on } \mathbb{C},$$

where g is a nonconstant entire function and $g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1 = k(|h(z_0)| + 1) + 1$. We claim that

$$(10) \quad \overline{E}_g(0) \subset \overline{E}_{g^{(k)}}(h(z_0)) \subset \overline{E}_{g^{(k+1)}}(0).$$

In fact, if there exists $\zeta_0 \in \mathbb{C}$, such that $g(\zeta_0) = 0$, then since $g(\zeta) \not\equiv 0$, there exist $\zeta_n, \zeta_n \rightarrow \zeta_0$, such that if n is sufficiently large,

$$g_n(\zeta_n) = \frac{f_n(z_n + \rho_n \zeta_n)}{\rho_n^k} = 0.$$

Thus $f_n(z_n + \rho_n \zeta_n) = 0$, so that $f_n^{(k)}(z_n + \rho_n \zeta_n) = h_n(z_n + \rho_n \zeta_n)$, i.e., that $g_n^{(k)}(\zeta_n) = h_n(z_n + \rho_n \zeta_n)$. Since $g^{(k)}(\zeta_0) = \lim_{n \rightarrow \infty} g_n^{(k)}(\zeta_n) = h(z_0)$, we have established the first part of the Claim that $\overline{E}_g(0) \subset \overline{E}_{g^{(k)}}(h(z_0))$.

Now, suppose there exists $\zeta_0 \in \mathbb{C}$, such that $g^{(k)}(\zeta_0) = h(z_0)$. If $g^{(k)}(\zeta) \equiv h(z_0)$, then $g^{(k+1)} \equiv 0$ and we are done. Thus we can assume that $g^{(k)}$ is not constant and since $f_n^{(k)}(z_n + \rho_n \zeta) - h_n(z_n + \rho_n \zeta) \Rightarrow g^{(k)}(\zeta) - h(z_0)$, we get by Hurwitz's Theorem that there exist $\zeta_n, \zeta_n \rightarrow \zeta_0$, such that

$$f_n^{(k)}(z_n + \rho_n \zeta_n) - h_n(z_n + \rho_n \zeta_n) = g_n^{(k)}(\zeta_n) - h_n(z_n + \rho_n \zeta_n) = 0.$$

Thus we have $f_n^{(k+1)}(z_n + \rho_n \zeta_n) = 0$ and $g_n^{(k+1)}(\zeta_n) = 0$. Letting $n \rightarrow \infty$, we get that $g^{(k+1)}(\zeta_0) = 0$. This completes the proof of the Claim. Now, by Lemmas 4 and 2, we have $g(\zeta) = \frac{h(z_0)(\zeta - \zeta_1)^k}{k!}$, where ζ_1 is a constant. Thus

$$g^\sharp(0) = \frac{|h(z_0)| |\zeta_1|^{k-1} / (k-1)!}{1 + |h(z_0)|^2 |\zeta_1|^{2k} / k!^2}.$$

Now, if $|\zeta_1| \leq 1$, then $g^\sharp(0) \leq \frac{|h(z_0)|}{(k-1)!} < kA + 1$, and if $|\zeta_1| > 1$, then $g^\sharp(0) \leq \frac{|h(z_0)||\zeta_1|^{k-1}/(k-1)!}{|h(z_0)|^2|\zeta_1|^{2k}/k!^2} \leq \frac{k \cdot k!}{|h(z_0)|} < kA + 1$. In either case we get a contradiction. \square

Similarly, we can get a slight generalization of Theorem CFZ3 for sequences.

Lemma 6. *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain $D \subset \mathbb{C}$, all of whose zeros are multiple and $\{h_n\}$ be a sequence of functions analytic on D such that $h_n(z) \Rightarrow h(z)$ on D , where $h(z) \neq 0$ for $z \in D$ and $s \geq 2$ be an even integer. Suppose that, for each n , $f_n(z) = 0 \Rightarrow f_n''(z) = h_n(z)$ and $f_n''(z) = h_n(z) \Rightarrow f_n'''(z) = f_n^{(s)}(z) = 0$, then $\{f_n\}$ is normal on D .*

The proof is very similar to the proof of Lemma 5. We start to argue the same (with 2 instead of k), and then instead of proving (10) we prove that

$$\overline{E}_g(0) \subset \overline{E}_{g''}(h(z_0)) \subset \overline{E}_{g^{(3)}}(0) \cap \overline{E}_{g^{(s)}}(0).$$

The left inclusion is proved in the same manner. Concerning the right inclusion, we now deduce from $f_n''(z_n + \rho_n \zeta_n) - h_n(z_n + \rho_n \zeta_n) = 0$ that $f_n^{(3)}(z_n + \rho_n \zeta_n) = f_n^{(s)}(z_n + \rho_n \zeta_n) = 0$. Then, since $\rho_n f_n^{(3)}(z_n + \rho_n \zeta) \Rightarrow g^{(3)}(\zeta)$ in \mathbb{C} and $\rho_n^{s-2} f_n^{(s)}(z_n + \rho_n \zeta) \Rightarrow g^{(s)}(\zeta)$ in \mathbb{C} , we conclude that $g^{(3)}(\zeta_0) = g^{(s)}(\zeta_0) = 0$. To get the final contradiction, we apply now Lemmas 4 and 3 instead of Lemmas 4 and 2.

The following result will play an essential role in treating transcendental functions which is used in the proofs of Theorems 2 and 3.

Theorem B. ([15] see also [2, p. 117]) *Let $f(z)$ be a function homomorphic in $\{z : R < |z| < \infty\}$, with essential singularity at $z = \infty$. Then $\overline{\lim}_{|z| \rightarrow \infty} |z|f^\sharp(z) = +\infty$.*

For the proof of Theorem 2, we need also the following Lemma.

Lemma 7. *Let h be a holomorphic function on D , with a zero of order $\ell(\geq 1)$ at $z_0 \in D$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions with zeros of multiplicity at least k , such that $\{f_n\}$ and h satisfy conditions (a) and (b) of Theorem 2. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of nonzero numbers such that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\{f_n(z_0 + \alpha_n \zeta)/\alpha_n^{k+\ell}\}_{n=1}^\infty$ is normal in \mathbb{C}^* .*

Proof. Without loss of generality, we may assume that $z_0 = 0$. In a neighborhood of the origin we have $h(z) = z^\ell b(z)$, where $b(z)$ is analytic, $b(0) \neq 0$. Define $r_n(\zeta) = \zeta^\ell b(\alpha_n \zeta)$.

We will show that the assumptions of Lemma 5 hold in \mathbb{C}^* for the sequence $\{G_n(\zeta)\}_{n=1}^\infty$, $G_n(\zeta) := f_n(\alpha_n\zeta)/\alpha_n^{k+\ell}$ and $\{r_n(\zeta)\}_{n=1}^\infty$. First, we have that $r_n(\zeta) \Rightarrow b(0)\zeta^\ell$ on \mathbb{C} and $\zeta^\ell \neq 0$ in \mathbb{C}^* . Assume that $G_n(\zeta) = 0$. Then $f_n(\alpha_n\zeta) = 0$ and $f_n^{(k)}(\alpha_n\zeta) = (\alpha_n\zeta)^\ell b(\alpha_n\zeta)$, and we get that $G_n^{(k)}(\zeta) = r_n(\zeta)$. Suppose now that $G_n^{(k)}(\zeta) = r_n(\zeta)$. This means that $f_n^{(k)}(\alpha_n\zeta) = h(\alpha_n\zeta)$ and thus $f_n^{(k+1)}(\alpha_n\zeta) = 0$. We have $G_n^{(k+1)}(\zeta) = 0$, and thus the assumptions of Lemma 5 hold. Hence we deduce that $\{G_n(\zeta)\}$ is normal in \mathbb{C}^* , and the lemma is proved. \square

The following lemma plays a similar role in the proof of Theorem 3, to the role of Lemma 7 in the proof of Theorem 2.

Lemma 8. *Let h be a holomorphic function on D , with a zero of order $\ell(\geq 1)$ at $z_0 \in D$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions whose zeros are multiple, such that $\{f_n\}$ and h satisfy conditions (a) and (b) of Theorem 3. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of nonzero numbers such that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\{f_n(z_0 + \alpha_n\zeta)/\alpha_n^{2+\ell}\}_{n=1}^\infty$ is normal in \mathbb{C}^* .*

The proof of this lemma is analogous to the proof of Lemma 7. Of course, we use Lemma 6 instead of Lemma 5.

3. PROOF OF THEOREM 1

In this section, we do not use any of the preliminary results. The proof is elementary.

By Theorem CFZ1, \mathcal{F} is normal at every point $z_0 \in D$ at which $h(z_0) \neq 0$ (so immediately we get that \mathcal{F} is quasinormal). So let z_0 be a zero of h of order $\ell(\geq 1)$. Without loss of generality, we can assume that $z_0 = 0$, and then $h(z) = z^\ell b(z)$. Here b is an analytic function in $\Delta(0, \delta)$ and $b(z) \neq 0$ there. We assume that $0 < \delta < 1$, and by taking a subsequence and renumbering, we can assume that

$$(11) \quad f_n \Rightarrow f \quad \text{in} \quad \Delta'(0, \delta).$$

Now, if f is holomorphic in $\Delta'(0, \delta)$, we deduce by the maximum principle that $f_n \Rightarrow f$ on $\Delta(0, \delta)$, and we are done. So let us assume that $f_n \Rightarrow \infty$ in $\Delta'(0, \delta)$. Fix η , $0 < \eta < \delta$. By the minimum principle (i.e., the maximum principle for $\{1/f_n\}$), there exists $N = N(\eta)$, such that for every $n \geq N$, f_n has $k_n(k_n \geq 1)$ simple zeros in

$\overline{\Delta}(0, \eta) - \{0\}$, say $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_{k_n}^{(n)}$ (otherwise we get that $f_n \Rightarrow \infty$ in $\Delta(0, \eta)$ and we are done). Since $f_n \Rightarrow \infty$ in $\Delta'(0, \delta)$, we get that

$$(12) \quad \max_{1 \leq j \leq k_n} \{|\alpha_j^{(n)}|\} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We can write $f_n(z) = t_n(z) \prod_{i=1}^{k_n} (z - \alpha_i^{(n)})$, where $t_n(z) \neq 0$ for $z \in \overline{\Delta}(0, \eta)$ and $n \geq N$.

Since $\eta < 1$, we get by (12) that $\frac{t_n(z)}{b(z)} \Rightarrow \infty$ in $\overline{\Delta}(0, \eta)$. By condition (a) of Theorem 1, we have, for $n \geq N$, $f'_n(\alpha_j^{(n)}) = \alpha_j^{(n)\ell} b(\alpha_j^{(n)})$, $1 \leq j \leq k_n$. By calculation,

$$f'_n(z) = t'_n(z) \prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) + t_n(z) \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]',$$

and so

$$(13) \quad t_n(\alpha_j^{(n)}) \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]' \Big|_{z=\alpha_j^{(n)}} = \alpha_j^{(n)\ell} b(\alpha_j^{(n)}).$$

Define, for $n \geq N$,

$$M_n(z) := \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]' - z^\ell.$$

By (13) we get that $M_n(\alpha_j^{(n)}) = 0$ for $1 \leq j \leq k_n$, and so for $n \geq N$, M_n has at least k_n zeros in $\Delta'(0, \eta)$, including multiplicities. Here we use the fact h has no common zero with any f_n . Since such a zero must be $z = 0$ and would be a zero of order m (must be $m \geq 2$ by condition (a)) of f_n , and it would be a zero of order $m - 1$ of M_n (if $\ell > m - 1$) or even of order $\ell < m - 1$ (if $\ell < m - 1$), then we would not know that the number of zeros (including multiplicities) of M_n is at least k_n . This fact, under the assumption that there are no common zeros, will lead to the desired contradiction.

Claim 2. $\frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]' \Rightarrow \infty$ in $\Delta'(0, \eta)$.

Proof. We write

$$(14) \quad \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]' = \sum_{j=1}^{k_n} \frac{t_n(z)}{b(z)} \prod_{i=1, i \neq j}^{k_n} (z - \alpha_i^{(n)}).$$

For any ε , $0 < \varepsilon < \eta$, we have that

$$(15) \quad \frac{t_n(z)}{b(z)} \prod_{i=2}^{k_n} (z - \alpha_i^{(n)}) \Rightarrow \infty \quad \text{in } \overline{R}_{\varepsilon, \eta} := \{z : \varepsilon \leq |z| \leq \eta\}.$$

Indeed, $\frac{t_n(z)}{b(z)} \prod_{i=2}^{k_n} (z - \alpha_i^{(n)}) = \frac{f_n(z)}{b(z)(z - \alpha_1^{(n)})}$, and since $\eta < 1$ and by (11) and (12), this term tends uniformly to ∞ in $\overline{R}_{\varepsilon, \eta}$.

Now, for every j , $2 \leq j \leq k_n$, we have that

$$\frac{\frac{t_n(z)}{b(z)} \prod_{i=2}^{k_n} (z - \alpha_i^{(n)})}{\frac{t_n(z)}{b(z)} \prod_{i=1, i \neq j}^{k_n} (z - \alpha_i^{(n)})} = \frac{z - \alpha_j^{(n)}}{z - \alpha_1^{(n)}},$$

and by (12) this term tends uniformly to 1 as $n \rightarrow \infty$. This means, that for every $1 \leq j \leq k_n$, and $z \in \overline{R}_{\varepsilon, \eta}$, $\frac{t_n(z)}{b(z)} \prod_{i=1, i \neq j}^{k_n} (z - \alpha_i^{(n)})$ lies in the same quarter plane, that is,

$$(16) \quad \Pi_{n,z} := \left\{ z : \arg \left[\frac{t_n(z)}{b(z)} \prod_{i=2}^{k_n} (z - \alpha_i^{(n)}) \right] - \frac{\pi}{4} < \arg z < \arg \left[\frac{t_n(z)}{b(z)} \prod_{i=2}^{k_n} (z - \alpha_i^{(n)}) \right] + \frac{\pi}{4} \right\},$$

for large enough n .

Now, if a and b are two complex numbers in the same quarter plane, then $a + b$ also belongs to that quarter plane and $|a + b| \geq |a|, |b|$. We then conclude by (16) that for each $z \in \overline{R}_{\varepsilon, \eta}$, we have for large enough n ,

$$\left| \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]' \right| \geq \left| \frac{t_n(z)}{b(z)} \prod_{i=2}^{k_n} (z - \alpha_i^{(n)}) \right|,$$

and by (15) and (14), the Claim is proved. \square

Now, $\frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]'$ has for large enough n exactly $k_n - 1$ zeros in $\Delta(0, \eta)$ (by Theorem Lu). Then for large enough n we have, for every z , $|z| = \eta$,

$$\left| M_n(z) - \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]' \right| = |z^\ell| < \left| \frac{t_n(z)}{b(z)} \left[\prod_{i=1}^{k_n} (z - \alpha_i^{(n)}) \right]' \right|,$$

and by Rouché's Theorem, we get that M_n has $k_n - 1$ zeros in $\Delta(0, \eta)$, a contradiction.

Theorem 1 is proved. \square

4. PROOF OF THEOREM 2

This proof is similar to the proof of Theorem 1 in [4]. By our Theorem 1, we need only to prove the case that $k \geq 3$. By Theorem CFZ2, \mathcal{F} is normal at every point

$z_0 \in D$ at which $h(z_0) \neq 0$ (so that \mathcal{F} is quasinormal in D). Consider $z_0 \in D$ such that $h(z_0) = 0$. Without loss of generality, we can assume that $z_0 = 0$, and then $h(z) = z^\ell b(z)$, where $\ell(\geq 1)$ is an integer, $b(z) \neq 0$ is an analytic function in $\Delta(0, \delta)$. We take a subsequence $\{f_n\}_1^\infty \subset \mathcal{F}$, and we want to prove that $\{f_n\}$ is not normal at $z = 0$. Suppose by negation that $\{f_n\}$ is not normal at $z = 0$. Since $\{f_n\}$ is normal in $\Delta'(0, \delta)$, we can assume (after renumbering) that $f_n \Rightarrow F$ on $\Delta'(0, \delta)$. If $F(z) \not\equiv \infty$, then it is a holomorphic function; hence by the maximum principle, F extends to be analytic also at $z = 0$, and so $f_n \Rightarrow F$ on $\Delta(0, \delta)$, and we are done. Hence we assume that

$$(17) \quad f_n(z) \Rightarrow \infty \quad \text{on} \quad \Delta'(0, \delta).$$

Define $\mathcal{F}_1 = \left\{ F = \frac{f_n}{h} : n \in \mathbb{N} \right\}$. It is enough to prove that \mathcal{F}_1 is normal in $\Delta(0, \delta)$. Indeed, if (after renumbering) $\frac{f_n(z)}{h} \Rightarrow H(z)$ on $\Delta(0, \delta)$, then since $h \neq 0$ in $\Delta'(0, \delta)$, it follows from (17) that $H(z) \equiv \infty$ in $\Delta'(0, \delta)$, and thus $H(z) \equiv \infty$ also in $\Delta(0, \delta)$. In particular, $\frac{f_n}{h}(z) \neq 0$ on each compact subset of $\Delta(0, \delta)$ for large enough n . Since $h \neq 0$ on $\Delta'(0, \delta)$ and since $f_n(0) \neq 0$ for every $n \geq 1$ by assumptions of the theorem, we obtain $f_n(z) \neq 0$ on each compact subset of $\Delta(0, \delta)$ for large enough n . Then by the minimum principle, it follows from (17) that $f_n(z) \Rightarrow \infty$ on $\Delta(0, \delta)$, and this implies the normality of \mathcal{F} . So suppose to the contrary that \mathcal{F}_1 is not normal at $z = 0$. By Lemma 1 and the assumptions of Theorem 2, there exist (after renumbering) points $z_n \rightarrow 0$, $\rho_n \rightarrow 0^+$ and a nonconstant meromorphic function on \mathbb{C} , $g(\zeta)$ such that

$$(18) \quad g_n(\zeta) = \frac{F_n(z_n + \rho_n \zeta)}{\rho_n^k} = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k h(z_n + \rho_n \zeta)} \xrightarrow{\chi} g(\zeta) \quad \text{on} \quad \mathbb{C},$$

all of whose zeros have multiplicity at least k and

$$(19) \quad \text{for every } \zeta \in \mathbb{C}, \quad g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1,$$

where $A > 1$ is a constant. Here we have used Lemma 1 with $\alpha = k$. Observe that $g_n(z) = 0$ implies $g_n^{(k)}(\zeta) = 1$ and so A can be chosen to be any number such that $A \geq 1$. After renumbering we can assume that $\{z_n/\rho_n\}_{n=1}^\infty$ converges. We separate now into two cases.

Case (A)

$$(20) \quad \frac{z_n}{\rho_n} \rightarrow \infty.$$

Claim 3. (1) $g(\zeta) = 0 \implies g^{(k)}(\zeta) = 1$; (2) $g^{(k)}(\zeta) = 1 \implies g^{(k+1)}(\zeta) = 0$.

Proof. Observe that from (18) and the fact that $h(z) \neq 0$ in $\Delta'(0, \delta)$, it follows that g is an entire function. Suppose that $g(\zeta_0) = 0$. Since $g(\zeta) \not\equiv 0$, there exist $\zeta_n \rightarrow \zeta_0$, such that $g_n(\zeta_n) = 0$, and thus $f_n(z_n + \rho_n \zeta_n) = 0$. Since f_n and h has no common zeros, it follows by the assumption that ζ_n is a zero of multiplicity k of $g_n(\zeta)$. By Leibniz's rule, and condition (a) of Theorem 2, it follows that $g_n^{(k)}(\zeta_n) = 1$ and thus $g^{(k)}(\zeta_0) = 1$.

For the proof of the other part of the Claim, observe first that by (20) we have

$$\frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k z_n^\ell} \Rightarrow g(\zeta) \quad \text{on } \mathbb{C},$$

and thus

$$\frac{f_n^{(k)}(z_n + \rho_n \zeta)}{z_n^\ell} \Rightarrow g^{(k)}(\zeta) \quad \text{on } \mathbb{C},$$

and then again by (19) we get that

$$\frac{f_n^{(k)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} \Rightarrow g^{(k)}(\zeta) \quad \text{on } \mathbb{C}.$$

Thus, if there exists $\zeta_0 \in \mathbb{C}$, such that $g^{(k)}(\zeta_0) = 1$, there exists a sequence $\zeta_n \rightarrow \zeta_0$, such that $f_n^{(k)}(z_n + \rho_n \zeta_n) = h(z_n + \rho_n \zeta_n) \neq 0$. By assumption (b) of Theorem 2 we get that $f_n^{(k+1)}(z_n + \rho_n \zeta_n) = 0$, and letting n tend to ∞ we get that $g^{(k+1)}(\zeta_0) = 0$. The Claim is proved. \square

We conclude by Lemma 2 and by Lemma 4 that $g(\zeta) = \frac{(\zeta - b)^k}{k!}$ for some $b \in \mathbb{C}$ (observe that g is holomorphic by (20)). By calculation we get that

$$g^\sharp(0) = \frac{|b|^{k-1}/(k-1)!}{1 + |b|^{2k}/k!^2}.$$

Then if $|b| \leq 1$, we get that $g^\sharp(0) \leq \frac{1}{(k-1)!}$, and if $|b| \geq 1$, then $g^\sharp(0) \leq \frac{k}{2}$. In either case, we get a contradiction to (19).

Case (B)

$$(21) \quad \frac{z_n}{\rho_n} \rightarrow \alpha \in \mathbb{C}.$$

As in Case (A), it follows that $g(\zeta_0) = 0 \implies g^{(k)}(\zeta_0) = 1$. Now set

$$G_n(\zeta) = \frac{f_n(\rho_n \zeta)}{\rho_n^{k+\ell}}.$$

From (18) and (21) we have

$$(22) \quad G_n(\zeta) \implies G(\zeta) = g(\zeta - \alpha)\zeta^\ell b(0) \quad \text{on } \mathbb{C}.$$

Indeed,

$$\frac{f_n(\rho_n \zeta)}{\rho_n^{k+\ell}} = \frac{f_n(\rho_n \zeta)}{\rho_n^k h(\rho_n \zeta)} \cdot \frac{h(\rho_n \zeta)}{\rho_n^\ell} = \frac{f_n\left(z_n + \rho_n\left(\zeta - \frac{z_n}{\rho_n}\right)\right)}{\rho_n^k h\left(z_n + \rho_n\left(\zeta - \frac{z_n}{\rho_n}\right)\right)} \frac{(\rho_n \zeta)^\ell b(\rho_n \zeta)}{\rho_n^\ell}$$

(cf. [12, p. 7]). Since g has a pole of order ℓ at $\zeta = -\alpha$ (here we use the fact that for every n , h has no common zeros with f_n) and since $\{G_n\}$ are analytic, we have

$$(23) \quad G(0) \neq 0, \infty.$$

We now consider several subcases, depending on the nature of G .

Case (BI) G is a polynomial.

Since $\{f_n\}$ is not normal at $z = 0$, there exist (after renumbering) a sequence $z_n^* \rightarrow 0$ such that

$$(24) \quad f_n(z_n^*) = 0.$$

Otherwise, there is some δ' , $0 < \delta' < \delta$ such that (before renumbering) $f_n(z) \neq 0$ in $\Delta(0, \delta')$, and since $f_n(z) \Rightarrow \infty$ on $\Delta'(0, \delta)$ we would have by the minimum principle that $f_n(z) \Rightarrow \infty$ on $\Delta(0, \delta)$, a contradiction to the non-normality of $\{f_n\}$ at $z = 0$. We have that all the zeros of g are of multiplicity exactly k . Then by (22) and (23), it follows that all the zeros of G are also of multiplicity exactly k . We consider now two possibilities.

Case (BI1) $\deg(G) = 0$.

We can assume that z_n^* from (24) is the closest zero of f_n to the origin. Then we have

$$(25) \quad \frac{f_n(\rho_n \zeta)}{\rho_n^{k+\ell} b(\rho_n \zeta)} \implies \frac{G(0)}{b(0)} \quad \text{on } \mathbb{C}.$$

By (25) we have

$$(26) \quad \frac{z_n^*}{\rho_n} \rightarrow \infty.$$

Define $t_n(\zeta) = f_n(z_n^* \zeta) / (z_n^{*k+\ell} b(z_n^* \zeta))$. We want to show that $\{t_n(\zeta)\}$ is normal in \mathbb{C}^* . For this purpose set $\tilde{t}_n(\zeta) = f_n(z_n^* \zeta) / z_n^{*k+\ell}$. Since $b(0) \neq 0, \infty$ and $z_n^* \rightarrow 0$, the normality of $\{t_n\}$ is equivalent to the normality of $\{\tilde{t}_n\}$, and the latter follows by

Lemma 7. Now, if $\{t_n\}$ is not normal at $\zeta = 0$, then we can write (after renumbering) $t_n(\zeta) \Rightarrow \infty$ on \mathbb{C}^* ; but $t_n(1) = 0$, so this is not possible. Hence $\{t_n(\zeta)\}$ is normal at $\zeta = 0$. By (25) and (26), $t_n(0) \rightarrow 0$ as $n \rightarrow \infty$; and thus since $t_n(\zeta) \neq 0$ in $\Delta(0, 1/2)$, we get by Hurwitz's Theorem that $t_n(\zeta) \Rightarrow 0$ on \mathbb{C} . But $t_n(1) = 0$; so by assumption (b) of Theorem 2, we get that $t_n^{(k)}(1) = 1$, a contradiction.

Case (BI2) $G^{(k)} \equiv b(0)\zeta^\ell$.

Then we have $G^{(k-1)}(\zeta) = \frac{b(0)\zeta^{\ell+1}}{\ell+1} + C$ and $G^{(k-2)}(\zeta) = \frac{b(0)\zeta^{\ell+2}}{(\ell+1)(\ell+2)} + C\zeta + D$, where C and D are two constants. Since all zeros of G have multiplicity exactly k , then for any zero $\hat{\zeta}$ of G , we have $G^{(k-2)}(\hat{\zeta}) = G^{(k-1)}(\hat{\zeta}) = 0$. So

$$(27) \quad \frac{\hat{\zeta}^{\ell+1}}{\ell+1} + C = 0, \quad \text{and} \quad \frac{\hat{\zeta}^{\ell+2}}{(\ell+1)(\ell+2)} + C\hat{\zeta} + D = 0.$$

By calculation, we have $\frac{(\ell+1)C}{\ell+2}\hat{\zeta} = -D$. If $CD = 0$, then by (27), $\hat{\zeta} = 0$, a contradiction. If $CD \neq 0$, then $\hat{\zeta} = -\frac{(\ell+2)D}{(\ell+1)C}$, which implies that G has only one zero ζ_0 , and then

$$G(\zeta) = \frac{b(0)\zeta_0^\ell(\zeta - \zeta_0)^k}{k!}.$$

This contradicts $G^{(k)} \equiv b(0)\zeta^\ell$.

Case (BI3) G is a nonconstant polynomial and $G^{(k)} \not\equiv b(0)\zeta^\ell$.

Since all zeros of G have multiplicity exactly k , we may assume that

$$G = A \prod_{j=1}^t (\zeta - \zeta_j)^k.$$

where $A \neq 0$ is a constant and $\zeta_j \neq 0$, $j = 1, 2, \dots, t$.

Claim 4. $G(\zeta) = 0 \implies G^{(k)}(\zeta) = b(0)\zeta^\ell \implies G^{(k+1)}(\zeta) = 0$.

Proof. Suppose first that $G(\zeta_0) = 0$. Then there exists a sequence, $\zeta_n \rightarrow \zeta_0$, such that $f_n(\rho_n \zeta_n) = 0$, and thus $f_n^{(k)}(\rho_n \zeta_n) = (\rho_n \zeta_n)^\ell b(\rho_n \zeta_n)$, that is, $\frac{f_n^{(k)}(\rho_n \zeta_n)}{\rho_n^\ell} = \zeta_n^\ell b(\rho_n \zeta_n)$. In the last equation, the left hand side tends to $\zeta_0^\ell b(0)$ as $n \rightarrow \infty$. This proves the first part of the Claim.

Suppose now that $G^{(k)}(\zeta_0) = b(0)\zeta_0^\ell$. Since $G^{(k)}(\zeta) \not\equiv b(0)\zeta^\ell$, there exists a sequence $\zeta_n \rightarrow \zeta_0$, such that $\frac{f_n^{(k)}(\rho_n \zeta_n)}{\rho_n^\ell} = \zeta_n^\ell b(\rho_n \zeta_n)$, that is, $f_n^{(k)}(\rho_n \zeta_n) = (\rho_n \zeta_n)^\ell b(\rho_n \zeta_n)$, and

thus $f_n^{(k+1)}(\rho_n \zeta_n) = 0$. Since $\frac{f_n^{(k+1)}(\rho_n \zeta)}{\rho_n^{\ell-1}} \Rightarrow G^{(k+1)}(\zeta)$, we deduce that $G^{(k+1)}(\zeta_0) = 0$, and this completes the proof of the Claim. \square

It follows from Claim 4 that $G^{(k+1)}(\zeta_j) = 0$, for $1 \leq j \leq t$.

If $t \geq 2$, we know that for every $1 \leq j \leq t$,

$$\begin{aligned} G^{(k+1)}(\zeta) &= A \left[\prod_{j=1}^t (\zeta - \zeta_j)^k \right]^{(k+1)} \\ &= A \left\{ \sum_{\mu=0}^{k+1} \binom{k+1}{\mu} [(\zeta - \zeta_j)^k]^{(k+1-\mu)} \left[\prod_{i=1, i \neq j}^t (\zeta - \zeta_i)^k \right]^{(\mu)} \right\} \\ &= A \left\{ (k+1)k! \left[\prod_{i=1, i \neq j}^t (\zeta - \zeta_i)^k \right]' + (\zeta - \zeta_j) P_j(\zeta) \right\}, \end{aligned}$$

where P_j is a polynomial. Thus, by Claim 4 we have

$$(28) \quad \left[\prod_{i=1, i \neq j}^t (\zeta - \zeta_i)^k \right]' \Big|_{\zeta_j} = 0, \quad 1 \leq j \leq t.$$

This means that for every $1 \leq j \leq t$,

$$\sum_{\substack{i=1 \\ i \neq j}}^t (\zeta - \zeta_j)^{k-1} \prod_{\substack{\ell=1 \\ \ell \neq i, j}}^t (\zeta - \zeta_\ell)^k \Big|_{\zeta_j} = 0.$$

Dividing in $\prod_{\ell \neq j} (\zeta_j - \zeta_\ell)^{k-1}$ gives

$$\sum_{\substack{i=1 \\ i \neq j}}^t \prod_{\substack{\ell=1 \\ \ell \neq i, j}}^t (\zeta_j - \zeta_\ell) = 0.$$

Thus $T''(\zeta_j) = 0$ for $1 \leq j \leq t$, where $T(\zeta) = \prod_{i=1}^t (\zeta - \zeta_i)$.

Now, if $t \geq 3$, then T'' is of degree $t-2$, and vanishes at t different points, a contradiction. If $t = 2$, we get from (28) that $\left[(\zeta - \zeta_2)^k \right]' \Big|_{\zeta_1} = 0$ and this is also a contradiction. So $t = 1$ and G has only one zero ζ_0 ($\zeta_0 \neq 0$), which means that $G(\zeta) = \frac{b(0)\zeta_0^\ell(\zeta - \zeta_0)^k}{k!}$.

By Hurwitz's Theorem, there exists a sequence $\zeta_{n,0} \rightarrow \zeta_0$, such that $G_n(\zeta_{n,0}) = 0$. If there exists δ' , $0 < \delta' < \delta$, such that for every n (after renumbering), $f_n(z)$ has only one zero $z_{n,0} = \rho_n \zeta_{n,0}$ in $\Delta(0, \delta')$.

Set

$$H_n(z) = \frac{f_n(z)}{(z - z_{n,0})^k}.$$

Since $H_n(z)$ is a nonvanishing holomorphic function in $\Delta(0, \delta')$ and $H_n(z) \Rightarrow \infty$ on $\Delta'(0, \delta)$, we can deduce as before by the minimum principle that $H_n(z) \Rightarrow \infty$ on $\Delta(0, \delta')$. But

$$H_n(2z_{n,0}) = \frac{f_n(2z_{n,0})}{z_{n,0}^k} = \frac{\rho_n^\ell G_n(2\zeta_{n,0})}{\zeta_{n,0}^k} \rightarrow 0,$$

a contradiction. Thus, we can assume, after renumbering, that for every $\delta' > 0$, f_n has at least two zeros in $\Delta(0, \delta')$ for large enough n . Thus, there exists another sequence of points $z_{n,1} = \rho_n \zeta_{n,1}$, tending to zero, where $z_{n,1}$ is also a zero of $f_n(z)$ and $\zeta_{n,1} \rightarrow \infty$, as $n \rightarrow \infty$. We can also assume that $z_{n,1}$ is the closest zero to the origin of f_n , except $z_{n,0}$. Now set $c_n = z_{n,0}/z_{n,1}$ and define $K_n(\zeta) = f_n(z_{n,1}\zeta)/z_{n,1}^{k+\ell}$. By Lemma 7, $\{K_n(\zeta)\}$ is normal in \mathbb{C}^* . Now, if $\{K_n\}$ is normal at $\zeta = 0$, then after renumbering we can assume that

$$K_n(\zeta) \Rightarrow K(\zeta) \quad \text{on } \mathbb{C}.$$

If $K(\zeta) \not\equiv \text{const.}$, then consider

$$L_n(\zeta) := \frac{K_n(\zeta)}{(\zeta - c_n)^k}.$$

Since $c_n \xrightarrow{n \rightarrow \infty} 0$, then the sequence $\{L_n\}_1^\infty$ is normal in \mathbb{C}^* . It is also normal at $\zeta = 0$. Indeed, $K_n(c_n) = 0$ (a zero of order k) and so L_n is a nonvanishing holomorphic function in $\Delta(0, 1)$. Thus (after renumbering)

$$L_n(\zeta) \Rightarrow \frac{K(\zeta)}{\zeta^k} \quad \text{on } \mathbb{C}.$$

But

$$L_n(0) = \frac{K_n(0)}{(-c_n)^k} = \frac{G_n(0)}{\zeta_{n,1}^\ell (-\zeta_{n,0})^k} \xrightarrow{n \rightarrow \infty} 0, \quad (\text{since } \zeta_{n,1} \xrightarrow{n \rightarrow \infty} \infty),$$

and $L_n(\zeta) \neq 0$ in $\Delta(0, 1/2)$; thus $K(\zeta)/\zeta^k \equiv 0$ in \mathbb{C} , a contradiction. If, on the other hand, $K(\zeta) \equiv \text{const.}$, then $K(\zeta) \equiv 0$ and $K^{(k)}(1) = 0$. But $K^{(k)}(1) = \lim_{n \rightarrow \infty} K_n^{(k)}(1) = \lim_{n \rightarrow \infty} \frac{f_n^{(k)}(z_{n,1})}{z_{n,1}^\ell} = \lim_{n \rightarrow \infty} \frac{h(z_{n,1})}{z_{n,1}^\ell} = \lim_{n \rightarrow \infty} b(z_{n,1}) = b(0)$, a contradiction. Hence we can deduce that $\{K_n\}$ is not normal at $\zeta = 0$, and since $K_n(\zeta)$ is holomorphic in Δ , then

$$K_n(\zeta) \Rightarrow \infty \quad \text{on } \mathbb{C}^*.$$

But $K_n(1) = 0$, a contradiction.

Case (BII) $G(\zeta)$ is a transcendental entire function.

Consider the family

$$\mathcal{F}(G) = \left\{ t_n(z) := \frac{G(2^n z)}{2^{n(k+\ell)}} : n \in \mathbb{N} \right\}.$$

By Claim 4, we deduce

- (i) $t_n(z) = 0 \implies t_n^{(k)}(z) = z^\ell$; and
- (ii) $t_n^{(k)}(z) = z^\ell \implies t_n^{(k+1)}(z) = 0$.

We then get by Theorem CFZ2 that $\mathcal{F}(G)$ is normal in \mathbb{C}^* . Thus there exists $M > 0$ such that for every $z \in R_{1,2} := \{z : 1 \leq |z| \leq 2\}$

$$t_n^\#(z) = \frac{2^{n(k+\ell+1)} |G'(2^n z)|}{2^{2n(k+\ell)} + |G(2^n z)|^2} \leq M.$$

Set $r(\zeta) := G(\zeta)/\zeta^{k+\ell}$. Then r is a transcendental meromorphic function, whose only pole is $\zeta = 0$. For every ζ , $|\zeta| \geq 2$ there exists $n \geq 1$ and $z \in R_{1,2}$, such that

$$(29) \quad \zeta = 2^n z.$$

Calculation gives

$$r^\#(\zeta) = \frac{|G'(\zeta)\zeta^{k+\ell} - (k+\ell)\zeta^{k+\ell-1}G(\zeta)|}{|\zeta|^{2(k+\ell)} + |G(\zeta)|^2}.$$

Thus, if $|\zeta| \geq 2$ satisfies (29) then

$$(30) \quad \begin{aligned} |\zeta r^\#(\zeta)| &= |2^n z| \frac{|G'(2^n z)(2^n z)^{k+\ell} - (k+\ell)(2^n z)^{k+\ell-1}G(2^n z)|}{|2^n z|^{2(k+\ell)} + |G(2^n z)|^2} \\ &\leq \frac{2^{k+\ell+1} \cdot 2^{n(k+\ell+1)} |G'(2^n z)|}{2^{2n(k+\ell)} + |G(2^n z)|^2} + \frac{(k+\ell)2^{(n+1)(k+\ell)} |G(2^n z)|}{2^{2n(k+\ell)} + |G(2^n z)|^2}. \end{aligned}$$

By separating into two cases, depending on $|G(2^n z)| > 2^{(n+1)(k+\ell)}$ or $|G(2^n z)| \leq 2^{(n+1)(k+\ell)}$, we see that the last expression in (30) is less or equal to

$$2^{k+\ell+1} t_n^\#(z) + (k+\ell)2^{2(k+\ell)}.$$

Thus, to every $|\zeta| \geq 2$,

$$|\zeta r^\#(\zeta)| \leq M \cdot 2^{k+\ell+1} + (k+\ell)2^{2(k+\ell)}.$$

But, according to Theorem B, $\overline{\lim}_{\zeta \rightarrow \infty} |\zeta r^\#(\zeta)| = \infty$, and we thus have a contradiction (cf. [3, pp. 19-21]). Theorem 2 is proved. \square

5. PROOF OF THEOREM 3

By Theorem CFZ3, \mathcal{F} is normal at every point $z_0 \in D$ at which $h(z_0) \neq 0$ (so that \mathcal{F} is quasinormal in D). Consider $z_0 \in D$ such that $h(z_0) = 0$. Without loss of generality, we can assume that $z_0 = 0$, and then $h(z) = z^\ell b(z)$, where $\ell(\geq 1)$ is an integer, $b(z) \neq 0$ is an analytic function in $\Delta(0, \delta)$. We take a subsequence $\{f_n\}_1^\infty \subset \mathcal{F}$, and we only need to prove that $\{f_n\}$ is not normal at $z = 0$.

Define $\mathcal{F}_2 = \left\{ F = \frac{f_n}{h} : n \in \mathbb{N} \right\}$. It is enough to prove that \mathcal{F}_2 is normal in $\Delta(0, \delta)$. Suppose to the contrary that \mathcal{F}_2 is not normal at $z = 0$. By Lemma 1 and the assumptions of Theorem 3, there exist (after renumbering) points $z_n \rightarrow 0$, $\rho_n \rightarrow 0^+$ and a nonconstant meromorphic function on \mathbb{C} , $g(\zeta)$ such that

$$(31) \quad g_n(\zeta) = \frac{F_n(z_n + \rho_n \zeta)}{\rho_n^2} = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^2 h(z_n + \rho_n \zeta)} \xrightarrow{\chi} g(\zeta) \quad \text{on } \mathbb{C},$$

all of whose zeros are multiple and

$$(32) \quad \text{for every } \zeta \in \mathbb{C}, \quad g^\sharp(\zeta) \leq g^\sharp(0) = 2A + 1,$$

where $A > 1$ is a constant. After renumbering we can assume that $\{z_n/\rho_n\}_{n=1}^\infty$ converges. We separate now into two cases.

Case (A) $\frac{z_n}{\rho_n} \rightarrow \infty$.

Similar to the proof of Theorem 2, we can prove that $g(\zeta) = 0 \implies g''(\zeta) = 1$ and that $g''(\zeta) = 1 \implies g'''(\zeta) = g^{(s)}(\zeta) = 0$. Then by Lemmas 4 and 3, we have

$$g(\zeta) = \frac{(\zeta - b)^2}{2},$$

for some $b \in \mathbb{C}$. Thus $g^\sharp(0) = \frac{|b|}{1 + |b|^4/4}$ and then $g^\sharp(0) \leq 1$, which contradicts (32).

Case (B)

$$(33) \quad \frac{z_n}{\rho_n} \rightarrow \alpha \in \mathbb{C}.$$

As in the proof of Theorem 2, we have $g(\zeta_0) = 0 \implies g''(\zeta_0) = 1$. Now set $G_n(\zeta) = \frac{f_n(\rho_n \zeta)}{\rho_n^{2+\ell}}$. From (31) and (33) we have

$$G_n(\zeta) \implies G(\zeta) = b(0)g(\zeta - \alpha)\zeta^\ell \quad \text{on } \mathbb{C}.$$

Since g has a pole of order ℓ at $\zeta = -\alpha$, $G(0) \neq 0, \infty$.

We now consider several subcases, depending on the nature of G .

Case (BI) G is a polynomial.

By a similar method of proof used in the proof of Theorem 2 (and using Lemma 8 instead of Lemma 7 in the appropriate places), we can get

$$G(\zeta) = \frac{b(0)\zeta_0^\ell(\zeta - \zeta_0)^2}{2},$$

and also we can arrive at a contradiction.

Case (BII) $G(\zeta)$ is a transcendental entire function.

Consider the family

$$\mathcal{F}(G) = \left\{ t_n(z) := \frac{G(2^n z)}{2^{n(2+\ell)}} : n \in \mathbb{N} \right\}.$$

We have

- (i) $t_n(z) = 0 \implies t_n''(z) = z^\ell$; and
- (ii) $t_n''(z) = z^\ell \implies t_n'''(z) = t_n^{(s)}(z) = 0$.

We then get by Theorem CFZ3 that $\mathcal{F}(G)$ is normal in \mathbb{C}^* . Set $r(\zeta) := G(\zeta)/\zeta^{2+\ell}$, and we have that, for every ζ , $|\zeta| \geq 2$, there exists $n \geq 1$ and $z \in R_{1,2}$, such that

$$|\zeta r^\sharp(\zeta)| \leq M \cdot 2^{2+\ell+1} + (2 + \ell)2^{2(2+\ell)}.$$

But, according to Theorem B, $\overline{\lim}_{\zeta \rightarrow \infty} |\zeta| r^\sharp(\zeta) = \infty$, and we thus have a contradiction (cf. [3, pp. 19-21]). Theorem 3 is proved. \square

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